Algebraic Geometry Final Examination, 2009, B.Math/M.Math/Ph.D.

Unless specified otherwise, you may assume the field k to be algebraically closed and of characteristic 0. Each question carries 12 marks. Anything proved in the class maybe cited without proof. Results of exercises, however, must be derived in full.

1. (i): Show that the element $4 + X$ in the formal power series ring $\mathbb{Z}[[X]]$ is irreducible.

   (ii): Show that the element $6 + X \in \mathbb{Z}[[X]]$ is reducible.

2. (i): Let k be a field and $K = k(X_1, .., X_n)$ be the field of rational functions in n variables over k. Show that K is not a k-algebra of finite type.

   (ii): Let X_0, X_1, X_2 denote homogeneous coordinates in $\mathbb{P}^2(k)$. Show that the set

   $$D(X_0X_1) \cup \{[1:0:0]\} \subset \mathbb{P}^2(k)$$

   is not a quasiprojective variety. This shows that the union of two quasiprojectives need not be quasiprojective. (*Hint:* A quasiprojective variety is open in its Zariski closure.)

3. (i): Let A be a Noetherian local ring with unique maximal ideal $\mathfrak{m}$, and let M be a finitely generated A-module. Suppose for some $k \geq 1$ we have:

   $m^k M$

   $$\mathfrak{m}^k M/\mathfrak{m}^{k+1} M = \mathfrak{m}^{k+1} M/\mathfrak{m}^{k+2} M$$

   Then show that $\cancel{M} = 0$.

   (ii): Prove that a plane projective curve of degree m can have at most $3m(m-2)$ points of inflexion.

4. Let $X = \mathbb{A}^1(k)$, and $Y = V(X_2^2 - X_1^3) \subset \mathbb{A}^2(k)$.

   (i): Show that the map $\phi : X \to Y$ defined by $\phi(t) = (t^2, t^3)$ is a bijective morphism. Is it an isomorphism? Justify your answer.

   (ii): Using the map ϕ above, describe the local ring $\mathcal{O}_{Y,0}$ (=the localisation of the the coordinate ring $\mathcal{O}(Y)$ at the maximal ideal $\mathfrak{m}_{(0,0)}$) as a subring of the field $k(t)$ of rational functions in one variable.

5. (i): Find the singular points and inflexion points of the cubic curve $V(X_0^3 + X_1^3 + X_0X_1X_2) \subset \mathbb{P}^2(k)$.

   (ii): Prove that the curve in (i) above is irreducible. Hence (or otherwise) prove that it is a rational curve.